Calculus 2 Problem Set 4 Answers January 2014

[1]

(1) $u = x - y, v = x + 2y$ とおいて $E : 0 \leq u \leq 1, 0 \leq v \leq 1$. $x = (2u + v)/3, y = (-u + v)/3$ より

$J(u, v) = 1/3$. 以上より

$$= \int_0^1 \int_0^1 3 \cdot \frac{1}{3}(2u + v) \cdot \frac{1}{3} \, du \, dv = \frac{1}{3} \int_0^1 \int_0^1 (2u + v) \, du \, dv = \frac{1}{3} \int_0^1 (1 + v) \, dv = \frac{1}{2}$$

(2) $u = x + y, v = x - y$ とおいて $E : 0 \leq u \leq 1, -1 \leq v \leq 1$. $x = (u + v)/2, y = (u - v)/2$ より

$J(u, v) = -1/2$. 以上より

$$= \int_{-1}^1 \int_0^1 u \cdot |-\frac{1}{2}| \, du \, dv = \frac{1}{2} \int_{-1}^1 \frac{1}{2} \, dv = \frac{1}{2}$$

(3) $u = x - y, v = x + y$ とおいて $E : 0 \leq u \leq \pi, 0 \leq v \leq \pi$. $x = (u + v)/2, y = (-u + v)/2$ より

$J(u, v) = 1/2$. 以上より

$$= \int_0^\pi \int_0^\pi u \sin v \cdot \frac{1}{2} \, du \, dv = \frac{\pi^2}{4} \int_0^\pi \sin v \, dv = \frac{\pi^2}{2}$$

[2] 指示された変数変換から

$$= \int_0^1 \int_0^1 uve^{-v} \cdot |-\frac{1}{2}| \, du \, dv = \frac{1}{4} \int_0^1 ve^{-v} dv = \frac{1}{4}\big(1 - \frac{2}{e}\big)$$

[3 指示された変数変換から $E : 0 \leq u \leq a, -u \leq v \leq u$.

$$= \int_0^a \int_{-u}^u e^{-u^2} \cdot |-\frac{1}{2}| \, du \, dv = \int_0^a ue^{-u^2} dv = \frac{1}{2}\big(1 - e^{-a^2}\big)$$

[4]

(1) $E : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 1$ だから

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos^2 \theta \, r \, dr \, d\theta = \frac{1}{4} \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = \frac{\pi}{8}$$

(2) $E : 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 4$ だから

$$= \int_0^{\pi/2} \int_0^4 \sqrt{16 - r^2} \, r \, dr \, d\theta = \frac{\pi}{2} \Big[\frac{2}{3}\big(-\frac{1}{2}\big)(16 - r^2)^{3/2}\Big]_0^4 = \frac{32\pi}{3}$$

(3) $E : 0 \leq \theta \leq 2\pi, 1 \leq r \leq 3$ だから

$$= \int_0^{2\pi} \int_1^3 \big(2r^2 \cos^2 \theta + 3r^2 \sin^2 \theta\big) \, r \, dr \, d\theta = 20 \int_0^{2\pi} (2 \cos^2 \theta + 3 \sin^2 \theta) \, d\theta = 100\pi$$

(4) $E : 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 1$ だから

$$= \int_0^{\pi/2} \int_0^1 r^2 \cos \theta \sin \theta e^{-r^2} \, r \, dr \, d\theta = \Big(\int_0^{\pi/2} \cos \theta \sin \theta \, d\theta\Big) \int_0^1 r^3 e^{-r^2} dr = \frac{1}{4}(1 - \frac{2}{e})$$

(5)　領域は講義で解説済み。 $E: 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\cos\theta$ だから

$$= \int_0^{\pi/2} \int_0^1 3r\sin\theta, r\,dr\;d\theta = \int_0^{\pi/2} 8\cos^2\theta \sin\theta\;d\theta = 2$$

[5]　$J(u,v) = 4uv, E: 0 \le u \le 1, 0 \le v \le 1-u$

$$= \int_0^1 \int_0^{1-u} v^2 \cdot 4uv\;dv\,du = \int_0^1 u(1-u)^4\,du = \frac{1}{30}$$

[6]　[4] と同様にすればよい。(4) 以外では領域は円盤、(4) では、積分の領域は３角形。

(1)　$D: 0 \le r \le 1, 0 \le \theta \le \pi, \displaystyle\int_0^{\pi} \int_0^1 r dr d\theta = \pi/2,$

(2)　$D: 0 \le r \le 1, 0 \le \theta \le \pi/2, \displaystyle\int_0^{\pi/2} \int_0^1 r^3 dr d\theta = \pi/8,$

(3)　$D: 0 \le r \le a, 0 \le \theta \le 2\pi, \displaystyle\int_0^{2\pi} \int_0^a r dr d\theta = \pi a^2,$

(4)　$D: \pi/4 \le \theta \le \pi/2, 0 \le r \le 6/\sin\theta, \displaystyle\int_{\pi/4}^{\pi/2} \int_0^{6/\sin\theta} r^2\cos\theta dr d\theta = 36,$

(5)　$D: 0 \le r \le 1, \pi \le \theta \le \dfrac{3}{2}\pi, \displaystyle\int_{\pi}^{3\pi/2} \int_0^1 \frac{2r dr d\theta}{1+r} = (1-\log 2)\pi,$

(6)　$D: 0 \le r \le \log 2, 0 \le \theta \le \pi/2, \displaystyle\int_0^{\pi/2} \int_0^{\log 2} e^r r dr d\theta = (2\log 2 - 1)(\pi/2),$

(7)　$D: 0 \le r \le 2\cos\theta, 0 \le \theta \le \pi/2, \displaystyle\int_0^{\pi/2} \int_0^{2\cos\theta} 3r^3\cos\theta\sin\theta dr d\theta = 2,$

(8)　$D: 0 \le \theta \le 2\pi, 0 \le r \le 1, \displaystyle\int_0^{2\pi} \int_0^1 \log(r^2+1) r dr d\theta = \pi(\log 4 - 1)$

[7]-[9] の曲線の概形は別紙参照。

[7]　$S = \displaystyle\int_0^{\pi/2} \int_0^{2(2-\sin 2\theta)^{1/2}} r dr d\theta = 2(\pi - 1)$

[8]　$S = 6\displaystyle\int_0^{\pi/6} \int_0^{12\cos\theta} r dr d\theta = 36\pi$

[9]　$S = \displaystyle\int_0^{\pi/2} \int_0^{1+\sin\theta} r dr d\theta = \frac{3\pi}{8} + 1$